On the principle of effective choice and its applications

Dedicated to
Professor Motokiti Kondô
on his sixtieth birthday anniversary

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Curiously enough, notwithstanding its original intention, there remains a lot of non-effective objects in descriptive set theory. For example, N. Lusin [9] proved that every non-enumerable Borel set has a regular parametric representation. The way by which one can define the representing mapping effectively, however, has been left uninvestigated.

In the present paper we shall show by some examples that almost all problems of effective definition of the above kind of objects can be solved by a unified process, which proceeds as follows: First, by making use of universal sets, a set of objects satisfying given requirements is turned into a set of functions. Then by choosing an element effectively from the latter we settle an effective definition of the object.

The validity of our method is greatly enlarged by a device which enables us to choose a point effectively from a Π_2^1 set of some category, which is worthwhile to note, since the principle of effective choice can not hold good for the Π_2^1 sets (Lévy [8], Third Theorem). The device was invented by the consideration of universal sets from the viewpoint of model theory and consists of the systematic use of Shoenfield's theorem and Addison's least function operator.

The usage of terminologies and notations is the same as in [13], except that the terminology "real number" is used to denote rational and irrational real numbers, while "function" is used to denote members of $\mathcal{N}(=N^N)$.

§1. Principle of effective choice

We shall first summarize several classical theorems concerning the effective choice of a point from a given non-void projective set [5, 10, 11, 12] in a list:

Given set	CA	$A_{\rho\sigma}$	B_2	P_2
Choice predicate	CA	$A_{\rho\sigma\delta}$	B_2	P_2

The corresponding uniformization theorems are as follows [5, 12]:

Given set	CA	$A_{\rho\sigma}$	B_2	P_2
Uniformizing set	CA	$A_{\rho\sigma\delta}$	B_2	P_2

The counterparts of these theorems in recursive function theory are as follows [5, 12, 15]:

Given set	Π_1^1	$(\Sigma_1^1)_{\rho\sigma}$	Δ_2^1	Σ_2^1
Choice predicate	Π_1^1	$(\Sigma_1^1)_{\rho\sigma\delta}$	Δ_2^1	Σ_2^1
Uniformizing set	Π_1^1	$(\Sigma_1^1)_{\rho\sigma\delta}$	Δ_2^1	Σ_2^1

It is well known that one can not find a general process of effective choice for Π_2^1 sets [8]. In some special cases, however, we can do it not only for Π_2^1 sets, but also for sets of higher classes.

To begin with we must modify Addison's lemma [2], as we do not assume the axiom of constructibility.

LEMMA 1. *Let $P(\mathfrak{A}, \beta)$ be a given predicate. On the condition that $\beta \in L$, we have*

$$\begin{aligned}(E\beta_1)_{\beta_1<\beta}P(\mathfrak{A}, \beta_1) &\equiv (E\beta_1)[(E\varphi)(E\varepsilon)[M(\varphi, \varepsilon) \ \& \ (Ei)[\omega\times\omega\cdot F`\varphi_i=\beta_1] \\ &\qquad \& \ \overline{(Ei)}[\omega\times\omega\cdot F`\varphi_i=\beta]] \ \& \ P(\mathfrak{A}, \beta_1)] \\ &\equiv (\varphi)(\varepsilon)[M(\varphi, \varepsilon) \ \& \ (Ei)[\omega\times\omega\cdot F`\varphi_i=\beta] \rightarrow (Ei)[(E\beta_2)[\omega\times\omega\cdot F`\varphi_i=\beta_2] \\ &\qquad \& \ (\beta_1)[\omega\times\omega\cdot F`\varphi_i=\beta_1 \rightarrow \beta_1\neq\beta \ \& \ P(\mathfrak{A}, \beta_1)]]]].\end{aligned}$$

PROOF. The first expression of $(E\beta_1)_{\beta_1<\beta}P(\mathfrak{A}, \beta_1)$ is the same and the second is a slight modification of the corresponding ones of Addison.

As an immediate consequence, we have the lemma:

LEMMA 2. *Let $E\subseteq\mathfrak{N}^k$ be a given set. On the assumption that $EL\neq\emptyset$, the effective choice of a point from the set E can be done by the following table:*

The set E	$\Sigma_n^1\ (n\geqq3)$ *or* $\Pi_n^1\ (n\geqq2)$	$\Delta_n^1\ (n\geqq2)$
Choice predicate	Δ_{n+1}^1 *(more precisely* $(\Sigma_n^1)_\rho$*)*	Δ_n^1

PROOF. For simplicity's sake, assume that $k=1$. Then

$$\alpha\in E \ \& \ \alpha\in L\mathfrak{N} \ \& \ \overline{(E\alpha_1)}_{\alpha_1<\alpha}[\alpha_1\in E]$$

is a choice predicate in which the second conjunct is Σ_2^1 (Addison [2]) and, by lemma 1, the third conjunct is in Π_n^1 or Σ_n^1 according as E is in Σ_n^1 or Π_n^1.

The uniformization principle in its full form does not hold for these sets (Lévy [8], First Theorem). Hence we need a new notion:

DEFINITION. Let E be a set in $\mathfrak{N}^k\times\mathfrak{N}$. By the *weakly uniformizing set* of E we mean a subset E_1 of E satisfying the following condition:

$$(\mathfrak{A})_L[(E\beta)_L[\langle\mathfrak{A}\beta\rangle \in E] \rightarrow (E!\beta)[\langle\mathfrak{A}\beta\rangle \in E_1]],$$

where $\mathfrak{A}$ is a variable on $\mathfrak{N}^k$.

Then we have the following lemma:

LEMMA 2′. *The weakly uniformizing set E_1 of a given set E can be obtained by the following table*:

Given set	$\Sigma_n^1 (n \geqq 3)$ *or* $\Pi_n^1 (n \geqq 2)$	$\Delta_n^1 (n \geqq 2)$
Weakly uniformizing set	Δ_{n+1}^1 (*more precisely* $(\Sigma_n^1)_\rho$)	Δ_n^1

PROOF. For short, suppose that $k=1$, then it suffices to define E_1 by

$$\langle\alpha\beta\rangle \in E_1 \equiv \langle\alpha\beta\rangle \in E \ \& \ \langle\alpha\beta\rangle \in \mathfrak{N}L \times \mathfrak{N}L \ \& \ \overline{(E\beta_1)}_{\beta_1<\beta}[\langle\alpha\beta_1\rangle \in E] .$$

§ 2. Enumerability

We begin the application of our method with the problem of effective enumeration of an enumerable A set[1)].

THEOREM 1. *The mapping, enumerating a given A, or Σ_1^1, set without repetition, can be defined by a B_2, or Δ_2^1, predicate.*

PROOF. For a $\beta \in \mathfrak{N}$, define a function $\beta^{\langle n\rangle}$ by

$$\beta^{\langle n\rangle}(m) = \beta(2^m(2n+1)-1).$$

Let $E \subseteq \mathfrak{N}$ be a given A, or Σ_1^1, set. Define the set E_1 by

$$\beta \in E_1 \equiv (m)(n)[m \neq n \rightarrow \beta^{\langle m\rangle} \neq \beta^{\langle n\rangle}] \ \& \ (n)[\beta^{\langle n\rangle} \in E] \\ \& \ (\alpha)[\alpha \in E \rightarrow (En)[\alpha = \beta^{\langle n\rangle}]].$$

E_1 is an A_ρ, or $(\Sigma_1^1)_\rho$, set. Let $P(\beta)$ be a B_2, or Δ_2^1, choice predicate. Then an enumerating mapping φ of the set E is defined by

$$\langle\alpha n\rangle \in \varphi \equiv (E\beta)[P(\beta) \ \& \ \beta^{\langle n\rangle} = \alpha] \equiv (\beta)[P(\beta) \rightarrow \beta^{\langle n\rangle} = \alpha].$$

COROLLARY. *Let E be a Σ_1^1 set. Then the set E and the function φ in theorem 1 are constructible in Gödel's sense.*

PROOF. By Shoenfield's theorem

$$(E\alpha)[\langle\alpha n\rangle \in \varphi] \equiv (E\alpha)(E\beta)[P(\beta) \ \& \ \beta^{\langle n\rangle} = \alpha] \equiv (E\alpha)_L(E\beta)_L[P_l(\beta) \ \& \ \beta^{\langle n\rangle} = \alpha] \\ \equiv (E\alpha)_L(E\beta)[P(\beta) \ \& \ \beta^{\langle n\rangle} = \alpha] \equiv (E\alpha)_L[\langle\alpha n\rangle \in \varphi]$$

for every n. On the other hand, by definition

1) This problem was first dealt with by M. Kondô [6]. Differing from ours, his method is geometrical. The effective enumerability was settled. However, no mention was made of the predicate which defines the enumerating mapping.

$$(E!\alpha)[\langle\alpha n\rangle\in\varphi].$$

Hence the function α such that $\langle\alpha n\rangle\in\varphi$ is constructible, and therefore $\varphi=\varphi\cdot L=\varphi_l$.

Since

$$\alpha\in E\equiv(En)[\langle\alpha n\rangle\in\varphi],$$

the set E consists only of constructible functions. Hence $E=EL=E_l$.

REMARK. The corollary shows that E_l is necessarily enumerable in the model Δ when E is enumerable in our universe.[1] The same can not hold for a Π_1^1 set. If E is in Π_1^1, then the cardinal of E_l in the model Δ depends not only on the cardinal of E in the universe, but also on the character of the universe. We shall return to this subject in section 5 again.

§ 3. Perfect sets

It is well known that an A set contains a perfect set if it is non-enumerable and a CA set contains a perfect set if a constituent of it is non-enumerable and conversely.

Let C be a fixed Cantor's discontinuum. We regard C as the set of all functions α^* such that $\alpha^*(n)=0$ or 1. A set $E\subseteq\mathfrak{N}$ contains a perfect set if and only if there exists a closed set $F\subseteq\mathfrak{N}\times C$ such that

(i) $(\alpha^*)(E\beta)[\langle\beta\alpha^*\rangle\in F]$,

(ii) $(\alpha^*)(\beta)[\langle\beta\alpha^*\rangle\in F\rightarrow\beta\in E]$,

(iii) $(\alpha^*)(\beta)(\beta_1)[\langle\beta\alpha^*\rangle\in F\ \&\ \langle\beta_1\alpha^*\rangle\in F\rightarrow\beta=\beta_1]$,

(iv) $(\alpha^*)(\alpha_1^*)(\beta)[\langle\beta\alpha^*\rangle\in F\ \&\ \langle\beta\alpha_1^*\rangle\in F\rightarrow\alpha^*=\alpha_1^*$,

(v) $$(\alpha^*)(\alpha_1^*)(m)(En)(\beta)(\beta_1)\Big[\langle\alpha^*\beta\rangle\in F\ \&\ \langle\alpha_1^*\beta_1\rangle\in F\ \&\ \operatorname{dis}(\alpha^*,\alpha_1^*)<\frac{1}{n}\rightarrow\operatorname{dis}(\beta,\beta_1)<\frac{1}{m+1}\Big],$$

where $\operatorname{dis}(\beta,\beta_1)$ is the usual metric in Baire's space:

$$\operatorname{dis}(\beta,\beta_1)=\frac{1}{k+1}\ \text{when}\ \beta(k)\neq\beta_1(k),\ \text{and}\ \beta(n)=\beta_1(n)\ \text{for all}\ n<k,$$

and $\operatorname{dis}(\alpha^*,\alpha_1^*)$ is its analogy in C:

$$\operatorname{dis}(\alpha^*,\alpha_1^*)=\frac{1}{k+1}\ \text{when}\ \alpha^*(k)\neq\alpha^*(k),\ \text{and}\ \alpha^*(n)=\alpha^*(n)\ \text{for all}\ n<k.$$

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1) By the *universe* we mean a model for set theory in which we develop the descriptive set theory and within which we construct the model Δ.

Consider a universal set $U \subseteq \mathfrak{N} \times C \times \mathfrak{N}$ for the family of the closed sets in $\mathfrak{N} \times C$. Then a set $F \subseteq \mathfrak{N} \times C$ is closed if and only if

$$(E\xi)[\langle \beta\alpha^* \rangle \in F \equiv \langle \beta\alpha^*\xi \rangle \in U].$$

Hence the existence of a closed set F satisfying the conditions (i)–(v) turns into that of a function ξ such that

$$(3.1;\xi) \qquad (\alpha^*)(E\beta)[\langle \beta\alpha^*\xi \rangle \in U],$$

$$(3.2;\xi) \qquad (\alpha^*)(\beta)[\langle \beta\alpha^*\xi \rangle \in U \rightarrow \beta \in E],$$

$$(3.3;\xi) \qquad (\alpha^*)(\beta)(\beta_1)[\langle \beta\alpha^*\xi \rangle \in U \ \& \ \langle \beta_1\alpha^*\xi \rangle \in U \rightarrow \beta = \beta_1],$$

$$(3.4;\xi) \qquad (\alpha^*)(\alpha_1^*)(\beta)[\langle \beta\alpha^*\xi \rangle \in U \ \& \ \langle \beta\alpha_1^*\xi \rangle \in U \rightarrow \alpha^* = \alpha_1^*],$$

$$(3.5;\xi) \qquad (\alpha^*)(\alpha_1^*)(m)(En)(\beta)(\beta_1)\Big[\langle \beta\alpha^*\xi \rangle \in U \ \& \ \langle \beta_1\alpha_1^*\xi \rangle \in U \ \& \ \mathrm{dis}\,(\alpha^*, \alpha_1^*) < \frac{1}{n} \rightarrow \mathrm{dis}\,(\beta, \beta_1) < \frac{1}{m+1}\Big],$$

where we may assume, as is well known, that the set U itself is closed.

Suppose that E is a CA set containing a perfect set. Then the set of all ξ's satisfying (3.1; ξ)–(3.5; ξ) is a non-void CA set. Hence we can choose a function ξ_0 from it by a CA predicate $P(\xi)$. The set $P = \hat{\beta}(E\alpha^*)[\langle \beta\alpha^*\xi_0 \rangle \in U]$ is perfect and contained in E. P is defined by a B_2 predicate, for

$$\beta \in P \equiv (E\xi)[P(\xi) \ \& \ (E\alpha^*)[\langle \beta\alpha^*\xi \rangle \in U]] \equiv (\xi)[P(\xi) \rightarrow (E\alpha^*)[\langle \beta\alpha^*\xi \rangle \in U]].$$

Thus we have the following theorem:

THEOREM 2. *A perfect set in a given CA set, when such exists, can be defined by a B_2 predicate.*

It is well known that a perfect set can be constructed in the least non-enumerable constituent of a CA set. With respect to this subject we have the following result.

COROLLARY 1. *A perfect set in the least non-enumerable constituent of a CA set can be defined by a B_2 predicate.*

PROOF. Let E be a CA set and let

$$E = \mathcal{E}_0 + \mathcal{E}_1 + \cdots + \mathcal{E}_\tau + \cdots \qquad (\tau < \Omega)$$

be its expansion by its constituents defined, for example, by a canonical sieve of Lebesgue type which defines the set E. Let ν be the function defined, as usual, by

$$\nu(\beta)=\begin{cases}\tau, & \beta\in\mathcal{E}_\tau,\\ \Omega, & \beta\notin E.\end{cases}$$

Consider the condition

$$(3.6;\xi)\qquad (E\gamma)(\alpha^*)(\beta)[\langle\beta\alpha^*\xi\rangle\in U\rightarrow(\delta)[\nu(\delta)+1\leqq\nu(\beta)\rightarrow(En)[\delta=\gamma^{\langle n\rangle}]]].$$

The set $\hat{\beta}(E\alpha^*)[\langle\beta\alpha^*\xi\rangle\in U]$ is contained in the least non-enumerable constituent if and only if $(3.1;\xi),\cdots,(3.5;\xi)$ and $(3.6;\xi)$ hold. The predicate $(3.1;\xi)$ & $\cdots$ & $(3.5;\xi)$ & $(3.6;\xi)$ is B_2, since, on the assumption of $(3.1;\xi)$–$(3.5;\xi)$, the condition $(3.6;\xi)$ is equivalent to the condition

$$(\xi_1)[(3.1;\xi_1)\ \&\ \cdots\ \&\ (3.5;\xi_1)\ \&\ (3.6;\xi_1)\rightarrow(\alpha^*)(\alpha_1^*)(\beta)(\beta_1)[\langle\beta\alpha^*\xi\rangle\in U\ \&\ \langle\beta_1\alpha_1^*\xi_1\rangle\in U\rightarrow\nu(\beta)=\nu(\beta_1)]].$$

Then the rest of the proof is carried out in the same way as above.

We shall call the least ordinal τ such that $\mathcal{E}_\tau$ is non-enumerable *Lusin's number*. By the *representative* of an ordinal number we mean a function ordering a set of numbers in such a way that the resulting ordered set represents the given ordinal number.

COROLLARY 2. *As a representative of Lusin's number of a CA set we can choose a function by a B_2 predicate.*

PROOF. It suffices to choose a function from the set P of Corollary 1.

The same method of defining a perfect set is applicable to the case where E is an A set. In this case, however, the condition $(3.2;\xi)$ turns into a CP_2 predicate, which demands the device used in §§ 6–7. Recently H. Tanaka obtained the result that one can define a perfect set in an A set by a B_2 predicate by considering details of the process of construction of the perfect set.

§ 4. Regular parametric representation

N. Lusin proved that *every non-enumerable B set has a regular parametric representation when one omits at most an enumerable number of points of it* ([9], p. 114). Our task in this section is to define the representing mapping of a set.

THEOREM 3. *A representing mapping of a non-enumerable B set can be defined by a B_2 predicate.*

PROOF. We identify the space $\mathfrak{N}$ with the set of all irrational numbers in the interval $(0,1)$ of the real line. Given a set $X\subseteq\mathfrak{N}$, denote by $X^\#$ the sum of sets X and $\{2,3,\cdots\}$. We introduce a metric into $\mathfrak{N}^\#$ by defining $\mathrm{dis}(\alpha^\#,\beta^\#)$ as usual when $\alpha^\#,\beta^\#\in\mathfrak{N}$ and putting $\mathrm{dis}(\alpha^\#,\beta^\#)=2$ otherwise.

Assume that the given B set E is contained in $\mathfrak{N}$. Let $U \subseteq \mathfrak{N}^{\#} \times \mathfrak{N}^{\#} \times \mathfrak{N}$ be a closed set which is universal for the family of all closed sets in $\mathfrak{N}^{\#} \times \mathfrak{N}^{\#}$. Then our first thing to do is to define effectively a ξ such that

$$(4.1; \xi) \qquad (\alpha^{\#})(E\beta^{\#})[\langle\beta^{\#}\alpha^{\#}\xi\rangle \in U \ \& \ \beta^{\#} \in E^{\#}],$$

$$(4.2; \xi) \qquad (\beta^{\#})[\beta^{\#} \in E^{\#} \rightarrow (E\alpha^{\#})[\langle\beta^{\#}\alpha^{\#}\xi\rangle \in U]],$$

$$(4.3; \xi) \qquad (\alpha^{\#})(\beta^{\#})(\beta_1^{\#})[\langle\beta^{\#}\alpha^{\#}\xi\rangle \in U \ \& \ \langle\beta_1^{\#}\alpha^{\#}\xi\rangle \in U \rightarrow \beta^{\#} = \beta_1^{\#}],$$

$$(4.4; \xi) \qquad (\alpha^{\#})(\alpha_1^{\#})(\beta^{\#})[\langle\beta^{\#}\alpha^{\#}\xi\rangle \in U \ \& \ \langle\beta^{\#}\alpha_1^{\#}\xi\rangle \in U \rightarrow \alpha^{\#} = \alpha_1^{\#}],$$

$$(4.5; \xi) \qquad (\alpha^{\#})(m)(En)(\alpha_1^{\#})(\beta^{\#})(\beta_1^{\#})\Big[\mathrm{dis}\,(\alpha^{\#}, \alpha_1^{*}) < \frac{1}{n} \ \& \ \langle\beta^{\#}\alpha^{\#}\xi\rangle \in U \ \& \ \langle\beta_1^{\#}\alpha_1^{\#}\xi\rangle \in U \rightarrow \mathrm{dis}\,(\beta^{\#}, \beta_1^{\#}) < \frac{1}{m+1}\Big].$$

On the assumption of $(4.3; \xi)$ and $(4.4; \xi)$, predicates $(4.1; \xi)$ and $(4.2; \xi)$ are equivalent to

$$(\alpha^{\#})(E!\beta^{\#})[\langle\beta^{\#}\alpha^{\#}\xi\rangle \in U \ \& \ \beta^{\#} \in E]$$

and

$$(\beta)[\beta \in E \rightarrow (E!\alpha^{\#})[\langle\beta\alpha^{\#}\xi\rangle \in U]]$$

respectively. Hence by Lusin's theorem concerning the projection of the unicity points of a B set (Lusin [9], p. 259[1]), each of $(4.1; \xi)$ and $(4.2; \xi)$ is equivalent to a CA predicate. Therefore we can define a ξ satisfying conditions $(4.1; \xi)$–$(4.5; \xi)$ by a CA predicate $P(\xi)$. Then the mapping φ defined by one of the following alternatives is a required one:

$$\langle\beta\alpha\rangle \in \varphi \equiv (E\xi)[P(\xi) \ \& \ \langle\beta\alpha\xi\rangle \in U] \equiv (\xi)[P(\xi) \rightarrow \langle\beta\alpha\xi\rangle \in U].$$

M. Kondô [7] obtained some results concerning the parametric representation. He proved, for example, that *every non-enumerable B set is a one-to-one continuous image of a given B set which does not reduce to an absolute F_σ set, when one omits at most an enumerable number of points from the former.* Our method is applicable to this case also.

Let M be a given B set not reducing to an absolute F_σ set and $E \subseteq \mathfrak{N}$ be a non-enumerable B set. Since a representing mapping of E on M can be extended to a continuous mapping on a G_δ set, it suffices to take, instead of U in $(4.1; \xi)$–$(4.5; \xi)$, a G_δ set in $\mathfrak{N}^{\#} \times \mathfrak{N}^{\#} \times \mathfrak{N}$ which is universal for the family of all G_δ sets in $\mathfrak{N}^{\#} \times \mathfrak{N}^{\#}$. Thus, as the conditions corresponding to $(4.1; \xi)$–$(4.5; \xi)$, we have

1) In his proof, unfortunately, the mapping just in question itself is made use of. There are other proofs adequate to our purpose. For example [16].

$$(4.1';\xi)\qquad (\alpha^{\#})[\alpha^{\#}\in M^{\#}\to(E\beta^{\#})[\langle\beta^{\#}\alpha^{\#}\xi\rangle\in U\ \&\ \beta^{\#}\in E^{\#}]],$$

$$(4.2';\xi)\qquad (\beta^{\#})[\beta^{\#}\in E^{\#}\to(E\alpha^{\#})[\langle\beta^{\#}\alpha^{\#}\xi\rangle\in U\ \&\ \alpha^{\#}\in M^{\#}]],$$

$$(4.3';\xi)\qquad (\alpha^{\#})(\beta^{\#})(\beta_1^{\#})[\langle\beta^{\#}\alpha^{\#}\xi\rangle\in U\ \&\ \langle\beta_1^{\#}\alpha^{\#}\xi\rangle\in U\to\beta^{\#}=\beta_1^{\#}],$$

$$(4.4';\xi)\qquad \begin{aligned}&(\alpha^{\#})(\alpha_1^{\#})(\beta^{\#})[\langle\beta^{\#}\alpha^{\#}\xi\rangle\in U\ \&\ \langle\beta^{\#}\alpha_1^{\#}\xi\rangle\in U\ \&\ \alpha^{\#}\in M^{\#}\\ &\qquad\qquad\qquad\qquad \&\ \alpha_1^{\#}\in M^{\#}\to\alpha^{\#}=\alpha_1^{\#}],\end{aligned}$$

$$(4.5';\xi)\qquad \begin{aligned}&(\alpha^{\#})(m)(En)(\alpha_1^{\#})(\beta^{\#})(\beta_1^{\#})\Big[\mathrm{dis}\,(\alpha^{\#},\alpha_1^{\#})<\frac{1}{n}\ \&\ \langle\beta^{\#}\alpha^{\#}\xi\rangle\in U\\ &\qquad\qquad \&\ \langle\beta_1^{\#}\alpha_1^{\#}\xi\rangle\in U\to\mathrm{dis}\,(\beta^{\#},\beta_1^{\#})<\frac{1}{m+1}\Big],\end{aligned}$$

where U is, as remarked above, a G_δ set. Hence the conjunction of $(4.1';\xi)$–$(4.5';\xi)$ is a CA predicate.

Thus we have the theorem:

THEOREM 4. *Kondô's representing mapping of a non-enumerable B set can be defined by a B_2 predicate.*

M. Kondô [5] proved also that *every non-enumerable A set, omitting at most an enumerable number of points of it, is a one-to-one continuous image of Lebesgue's CA set.* In order to deal with this problem in just the same way as above, we must choose a point effectively from a CP_3 set, since $(4.2';\xi)$ turns into a CP_3 predicate when M is a CA set, which disallows us to apply the device in §§ 6—7. A closer analysis of Kondô's construction of the representing mapping of course leads us to the answer, which is so complicated that it is inconvenient to include here.

§ 5. Invariance

In solving a problem in descriptive set theory, when the use of constructible functions is needed, we can never help noticing the importance of a sort of unaffectedness of meaning of set-theoretical concepts. In fact, in the previous paper [13] it was shown that the notion of equality is invariant for the Δ_2^1 sets irrespective of whether it is considered in our universe or in the model Δ, which played there a fundamental rôle. Here again we must try to find out useful ideas of this sort.

Let

$$(*)\qquad \delta_0,\ \delta_1,\ \cdots,\ \delta_n,\ \cdots$$

be a recursive enumeration of Baire's intervals, that is, let a primitive recursive predicate $R(\alpha, x)$ such that $\alpha\in\delta_n\equiv R(\alpha, n)$ exist.

Define a set $U\subseteq\mathfrak{N}\times\mathfrak{N}$ by

$$\langle\alpha\xi\rangle\in U\equiv(En)[\alpha\in\delta_{\xi(n)}]$$

As is well known, the set U not only itself is open in $\mathfrak{N}\times\mathfrak{N}$, but also is universal for the family of all open sets in $\mathfrak{N}$. We shall show that these properties of U are unaffected although U is considered as a set in the model Δ.

Let $U_l=UL$. Since $R(\bar{\alpha}, x)$[1] is primitive recursive in the model Δ, the collection of all $\langle\bar{\alpha}\bar{\xi}\rangle$'s such that $(En)[R(\bar{\alpha}, \bar{\xi}(n))]$ is a set in the model Δ. Let it be $\bar{U}$. Then $(\bar{\alpha})(\bar{\xi})[\langle\bar{\alpha}\bar{\xi}\rangle\in U_l\equiv\langle\bar{\alpha}\bar{\xi}\rangle\in\bar{U}]$. Hence, taking into account that both U_l and $\bar{U}$ are contained in $\mathfrak{N}L\times\mathfrak{N}L$, we conclude that $U_l=\bar{U}\in L$.

Since $R(\bar{\alpha}, x)$ is recursive in the model Δ, by Addison's fundamental lemma[2] concerning the relationship between both open and closed sets and recursive predicates [1], $\bar{U}$ is open in the model Δ. Hence $U_l=\bar{U}$ is open in the model Δ. Therefore the set $U_l^{\langle\bar{\xi}\rangle}=\hat{\bar{\alpha}}[\langle\bar{\alpha}\bar{\xi}\rangle\in U_l]$, $\bar{\xi}\in\mathfrak{N}L$, is open in the model Δ.

Conversely, suppose that $\bar{G}$ is an open set in the model Δ. Then there is a subsequence of $(*)$

$$(**)\qquad \delta_{n_0}, \delta_{n_1}, \cdots, \delta_{n_k}, \cdots$$

such that $\bar{\alpha}\in\bar{G}\equiv(Ek)[\bar{\alpha}\in\delta_{n_k}]$. Since $(**)$ is a sequence in the model Δ, there is a $\bar{\xi}\in\mathfrak{N}L$ such that $\bar{\xi}(k)=n_k$ for every k. Hence

$$\bar{\alpha}\in\bar{G}\equiv(Ek)[\bar{\alpha}\in\delta_{n_k}]\equiv(Ek)[\bar{\alpha}\in\delta_{\bar{\xi}(k)}]\equiv\langle\bar{\alpha}\bar{\xi}\rangle\in U_l.$$

Therefore, there is a $\bar{\xi}$ such that $\bar{G}=U_l^{\langle\bar{\xi}\rangle}$.

Thus, it is shown that the set U_l is open in the model Δ and universal there for the family of all open sets in the model Δ.

As an immediate consequence it follows that there exists a universal set with the same properties for all closed, or G_δ, or F_σ, sets. We shall call them *invariant universal sets.*

As an illustration of the importance of the invariant universal set, we prove that *a Π_1^1 set E contains a perfect set if and only if the same holds for $E_l\,(=EL)$ in the model Δ.*

The set E contains a perfect set if and only if $(E\xi)[(3.1;\xi)\;\&\cdots\&\;(3.5;\xi)]$. By Shoenfield's theorem, we have

$$(E\xi)[(3.1;\xi)\;\&\cdots\&\;(3.5;\xi)]\equiv(E\xi)_L[(3.1;\xi)_l\;\&\cdots\&\;(3.5;\xi)_l],$$

where $(3.i;\xi)_l$ denotes the predicate resulting from $(3.i;\xi)$ by restricting all quantifiers to L and by replacing U and E by U_l and E_l respectively $(i=1, 2, \cdots 5)$. Hence, the invariance of the universal set U immediately implies our assertion.

We can prove the same for a Σ_1^1 set E. The same method holds

1) Dashes denote variables in the model Δ.

2) T. Tugué was also aware of the same fact independently from J. W. Addison.

good in this case also. However, there is a short cut here:

$$(E \text{ contains a perfect set}) \equiv (E \text{ is non-enumerable})$$
$$\equiv (E_l \text{ is non-enumerable in } \varDelta) \equiv (E_l \text{ contains a perfet set in } \varDelta),$$

where the second equivalence is an immediate consequence of the corollary of theorem 1.

It is remarkable that the invariance of enumerability does not hold for Π_1^1 sets, contrary to Σ_1^1 sets. In fact, the implication

(***) $\qquad (E \textit{ is enumerable}) \longrightarrow (E_l \textit{ is enumerable in } \varDelta)$

is undecidable in ZF for a Π_1^1 *set* E_1, though the converse, that is,

(****) $\qquad (E_l \textit{ is enumerable in } \varDelta) \longrightarrow (E \textit{ is enumerable})$

holds good.

The assertion (***) is not refutable, since it is true when $V=L$ holds in our universe. In order to show that it is not provable, assume that in our universe every definable well ordering of real numbers is enumerable or finite (Lévy [8], Second Theorem). By Addison [2] and Kondô [5], $\mathfrak{N}L$ is the projection of a uniform Π_1^1 set E. E is enumerable in the universe, since the cardinal of E is the same as that of $\mathfrak{N}L$ and the latter is well ordered. By Shoenfield's theorem, E contains only constructible points, hence $E=E_l$. E_l is non-enumerable in the model $\varDelta$, since its cardinal is the same as that of $\mathfrak{N}L$ in the model $\varDelta$ and the latter is non-enumerable there.

Now we prove (****). Making use of Shoenfield's theorm, we have

$$(E_l \text{ is enumerable in } \varDelta) \equiv (E\alpha)_L(\beta)_L[\beta \in E_l \longrightarrow (En)[\beta = \alpha^{\langle n \rangle}]]$$
$$\equiv (E\alpha)_L(\beta)[\beta \in E \longrightarrow (En)[\beta = \alpha^{\langle n \rangle}]] \Rightarrow (E\alpha)(\beta)[\beta \in E \longrightarrow (En)[\beta = \alpha^{\langle n \rangle}]]$$
$$\equiv (E \text{ is enumerable}).$$

As a corollary to (***), we have that *it is undecidable in ZF whether the set* $\mathfrak{N}L$ *is non-enumerable or not.*

§ 6. Measure

N. Lusin and W. Sierpiński proved that an A set is measurable in Lebesgue's sense. In this section we shall deal with the property of the measure of a Σ_1^1 set and related problems.

THEOREM 5. *The measure of a* Σ_1^1 *set is a constructible real number which can be defined by a* $\varDelta_3^1$ *predicate. A* G_δ *set including the given* Σ_1^1 *set and having the same measure with it can also be defined by a* $\varDelta_3^1$ *predicate.*

PROOF. Let $E \subseteq \mathfrak{N}$ be a given Σ_1^1 set and $U \subseteq \mathfrak{N} \times \mathfrak{N}$ be an invariant universal set of the family of all G_δ sets in $\mathfrak{N}$. The defining postulate of the outer measure of E is

$$(6.1;\ \eta^b) \qquad (\xi)[(\alpha)[\alpha \in E \to \langle \alpha\xi \rangle \in U] \to \eta^b \leqq \mu_U\langle \xi \rangle]] \\ \& \ (E\xi)[(\alpha)[\alpha \in E \to \langle \alpha\xi \rangle \in U] \ \& \ \eta^b = \mu_U\langle \xi \rangle]],$$

where $\mu_U\langle \xi \rangle$ is an arithmetical functional calculating the measure of the set $\hat{\alpha}[\langle \alpha\xi \rangle \in U]$ and η^b is a variable on the interval [0, 1] of the real line.

Let $\bar{\eta}_0^b$ be the outer measure of E_l in the model Δ. Then, since U_l is a universal set of all G_δ sets in the model Δ, $\bar{\eta}_0^b$ satisfies $(6.1;\ \bar{\eta}^b)_l$. Shoenfield's theorem can be applied to the predicate $(6.1;\ \eta^b)$, since it is the conjunction of a Π_2^1 predicate and a Σ_2^1 predicate. Hence we have that $(6.1;\ \bar{\eta}_0^b)_l \equiv (6.1;\ \bar{\eta}_0^b)$ and consequently that $(6.1;\ \bar{\eta}_0^b)$ and finally that $\bar{\eta}_0^b$ is the outer measure of E, proving the constructibility of the measure of a Σ_1^1 set.

Consider the condition:

$$(6.2;\ \xi;\ \eta^b) \qquad (\alpha)[\alpha \in E \to \langle \alpha\xi \rangle \in U] \ \& \ \eta^b = \mu_U\langle \xi \rangle.$$

Since $(6.1;\ \bar{\eta}_0^b)$ holds good, there is a function ξ satisfying $(6.2;\ \xi;\ \bar{\eta}_0^b)$. Uniformize the ξ's satisfying $(6.2;\ \xi,\ \eta^b)$ by a Π_1^1 predicate $P(\xi, \eta^b)$. Then a G_δ set G required in the theorem is defined by

$$\alpha \in G \equiv (E\xi)(E\eta^b)[(6.1;\ \eta^b) \ \& \ P(\xi;\ \eta^b) \ \& \ \langle \alpha\xi \rangle \in U] \\ \equiv (\xi)(\eta^b)[(6.1;\ \eta^b) \ \& \ P(\xi, \eta^b) \to \langle \alpha\xi \rangle \in U].$$

Recently H. Tanaka improved this result. We present here, however, our original result and proof, since the proof seems to be an appropriate illustration of our method.

Let E_1 be a CA set and

$$E_1 = \mathcal{E}_0 + \mathcal{E}_1 + \cdots + \mathcal{E}_\tau + \cdots$$

be an expansion of it by its constituents. It is shown by Sélivanowski that there exists an ordinal number τ such that

$$m(\mathcal{E}_\tau + \cdots) = 0.$$

Let us call the least ordinal of such τ's *Sélivanowski's number*. Our next task is to define it.

We assume here that E_1 is a Π_1^1 set and, for brevity, assume moreover that it is the complement of the set E in theorem 5. The condition which should be satisfied by the representative α of Sélivanowski's number is

$$m\Big(\sum_{\tau < \nu(\alpha)} \mathcal{E}_\tau\Big) = 1 - \bar{\eta}_0^b \ \& \ (m)(En)[\nu_m(\alpha) \leqq \nu_n(\alpha) \ \& \ m(\mathcal{E}_{\nu_n(\alpha)}) > 0],$$

where $\nu_m(\alpha)$ denotes the order type of the m section of the given sieve[1] at α. This is a predicate in $\Delta^1_2[\bar{\eta}^\flat_0]$. Hence we have the theorem:

THEOREM 6. *A representative of Sélivanowski's number of a* Π^1_1 *set can be defined by a* Δ^1_3 *predicate.*

§ 7. Category

LEMMA 3. *A set* E *has Baire's property if and only if there are an open set* G *and two sets* S, T *of the first category such that*

$$E \subseteq G+S \quad \textit{and} \quad G \subseteq E+T.$$

PROOF. Suppose that E has Baire's property. Then it has the form

$$E=G-P+R.$$

where G is an open set and P, R are sets of the first category. Hence

$$E \subseteq G+R \quad \text{and} \quad G-P \subseteq E.$$

From the latter we have

$$G \subseteq E+P.$$

Conversely, suppose that E satisfies the condition in this lemma. Then

$$E=EG+ES=G-(G-E)+ES.$$

Put $P=G-E$ and $R=ES$, then $E=G-P+R$. Since $P \subseteq T$ and $R \subseteq S$, P and R are of the first category.

LEMMA 3′. *In the condition of lemma* 3 *we can replace the sets* S *and* T *respectively by* Q_1 *and* Q_2 *which are not only of the first category but also of the class* F_σ.

PROOF. By the definition of the first category we have F_σ sets Q_1 and Q_2 of the first category such that $S \subseteq Q_1$ and $T \subseteq Q_2$. Hence

$$E \subseteq G+Q_1 \quad \text{and} \quad G \subseteq E+Q_2.$$

The converse is obvious.

N. Lusin and W. Sierpiński showed that an A set has Baire's property, or more precisely, Baire's property in the strict sense. For simplicity's sake, we confine ourselves to the former property.

THEOREM 7. *Given a* Σ^1_1 *set* E *we can define three sets* G, Q_1 *and* Q_2 *in lemma* 3′ *by* Δ^1_3 *predicates.*

1) For the terminology "the section of a sieve", cf. [11].

PROOF. Suppose that $E \subseteq \mathfrak{N}$. Let U and W be universel sets respectively for all open sets and for all F_σ sets of the first category. The construction of the latter differs from the universal set for all F_σ sets, mentioned above in § 5, only if we impose some arithmetical conditions; hence we may assume that it is invariant. The same is of course assumed for the former.

The set E has Baire's property if and only if there is a $\langle \xi\eta_1\eta_2 \rangle$ satisfying the following condition:

$$(7.1;\ \xi,\eta_1,\eta_2) \qquad \begin{aligned} &(\alpha)[\alpha \in E \rightarrow \langle \alpha\xi \rangle \in U \vee \langle \alpha\eta_1 \rangle \in W] \\ &\qquad \& \ (\alpha)[\langle \alpha\xi \rangle \in U \rightarrow \alpha \in E \vee \langle \alpha\eta_2 \rangle \in W]. \end{aligned}$$

Since E_l is an A set in the model Δ, by Lusin-Sierpiński's theorem, it has Baire's property in the model Δ. That is, there is a $\langle \bar{\xi}\bar{\eta}_1\bar{\eta}_2 \rangle$ satisfying the condition $(7.1;\ \bar{\xi}, \bar{\eta}_1, \bar{\eta}_2)_l$. Let $\langle \bar{\xi}_0\bar{\eta}_{10}\bar{\eta}_{20} \rangle$ be an element with this property. Then $(7.1;\ \bar{\xi}_0, \bar{\eta}_{10}, \bar{\eta}_{20})_l$ holds good. On the other hand, by Shoenfield's theorem, $(7.1;\ \bar{\xi}_0, \bar{\eta}_{10}, \bar{\eta}_{20}) \equiv (7.1;\ \bar{\xi}_0, \bar{\eta}_{10}, \bar{\eta}_{20})_l$. Hence $(7.1;\ \bar{\xi}_0, \bar{\eta}_{10}, \bar{\eta}_{20})$ also holds good, showing that there are constructible $\langle \xi\eta_1\eta_2 \rangle$'s satisfying $(7.1;\ \xi, \eta_1, \eta_2)$. Hence, by lemma 2, we can choose one of them by a Δ_3^1 predicate $P(\xi, \eta_1, \eta_2)$. Then the required G, Q_1 and Q_2 are defined by

$$\begin{aligned} \alpha \in G &\equiv (E\xi)(E\eta_1)(E\eta_2)[P(\xi,\eta_1,\eta_2) \ \& \ \langle \alpha\xi \rangle \in U] \\ &\equiv (\xi)(\eta_2)(\eta_2)[P(\xi,\eta_1,\eta_2) \rightarrow \langle \alpha\xi \rangle \in U], \end{aligned}$$

and

$$\begin{aligned} \alpha \in Q_i &\equiv (E\xi)(E\eta_1)(E\eta_2)[P(\xi,\eta_1,\eta_2) \ \& \ \langle \alpha\eta_i \rangle \in W] \\ &\equiv (\xi)(\eta_1)(\eta_2)[P(\xi,\eta_1,\eta_2) \rightarrow \langle \alpha\eta_i \rangle \in W] \end{aligned}$$

$(i=1, 2)$.

References

[1] Addison, J. W., *Separation principles in the hierarchies of classical and effective descriptive set theory*, Fund. Math., **46** (1959), pp. 123-136.

[2] ———, *Some consequences of the axiom of constructibility*, Fund. Math., **46** (1959), pp. 337-357.

[3] Gödel, K., *The consistency of the axiom of choice and of the generalized continuum-hypothesis with the axioms of set theory*, Princeton University, 1951.

[4] Kondô, M., *Sur la représentation paramétrique régulière des ensembles analytiques*, Fund. Math., **31** (1938), pp. 29-46.

[5] ———, *Sur l'uniformization des complémentaires analytiques et les ensembles projectifs de la seconde classe*, Jap. J. Math., **15** (1938), pp. 197-230.

[6] ———, *On enumerable sets* (in Japanese), Tokyo Butsuri Gakkô Zassi, No. **577** (1939), pp. 50-55.

[7] ———, *Sur la représentation paramétrique des ensembles*, J. Fac. Sci. Hokkaido Imp. Univ., **8** (1940), pp. 173-220.

[8] Lévy, A., (no title), mimeographed notes, 1963.

[9] Lusin, N., *Leçons sur les ensembles analytiques et leurs applications*, Gauthier-Villars, 1930.

[10] Lusin, N. & Novikoff, P., *Choix effectif d'un point dans un complémentaire analytique arbitraire donné par un crible*, Fund. Math., **25** (1935), pp. 559-560.

[11] Sampei, Y., *Note on the effective choice of a point in the complement of an analytic set*, Comment. Math. Univ. St. Paul., **9** (1961), pp. 91-95.

[12] ———, *On the uniformization of a set of class $A_{\rho\sigma}$*, Comment. Math. Univ. St. Paul., **10** (1962), pp. 67-73.

[13] ———, *On the complete basis of the Δ_2^1 sets*, Comment. Math. Univ. St. Paul., **13** (1965), pp. 81-88.

[14] Shoenfield, J. W., *The problem of predicativity*, Essays on the Foundations of Mathematics, Jerusalem, 1961, pp. 132-139.

[15] Suzuki, Y., *On the uniformization principle*, Proceedings of the Symposium on the Foundations of Mathematics, Held at Katada, Tokyo, 1962, pp. 137-144.

[16] Tugué, T. & Tanaka, H., *A note on the effective descriptive set theory*, to appear in these Commentarii.

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